Darboux transformations for 5-point and 7-point self-adjoint schemes and an integrable discretization of the 2D Schrödinger operator

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February 8, 2008

Abstract

With this paper we begin an investigation of difference schemes that possess Darboux transformations and can be regarded as natural discretizations of elliptic partial differential equations. We construct, in particular, the Darboux transformations for the general self adjoint schemes with five and seven neighbouring points. We also introduce a distinguished discretization of the two-dimensional stationary Schrödinger equation, described by a 5-point difference scheme involving two potentials, which admits a Darboux transformation.

Keywords: Darboux Transformations, Difference Equations, Lattice Integrable Nonlinear σ-models

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1 Introduction

Linear differential operators admitting Darboux Transformations (DTs) play a crucial role in the theory of integrable systems [1, 2]. i) One can associate with such operators integrable nonlinear partial differential equations (PDEs). ii) One can make use of the spectral theory of such linear operators to solve classical initial - boundary value problem for the nonlinear PDEs. iii) One can construct solutions of the nonlinear equations from simpler solutions, using the Darboux - Bäcklund transformations. iv) One can often associate with the nonlinear PDE a geometric meaning, inherited by the geometric properties of the linear operator.

It is natural to search for a discretization of this beautiful picture. In general, searching for distinguished discretizations of linear differential operators that admit general DTs (integrable discretizations) is not a trivial task. Several methods of "integrable discretization" have been used so far (see e.g. [3]), but no one is fully satisfactory. Surprisingly enough, a distinguished integrable discretization of the two-dimensional stationary Schrödinger equation is, to the best of our knowledge, not present in the literature and one of our goals is to change this situation.

In recent years the study of linear difference equations that admit DTs were undertaken. Most of the results are based on the 4-point difference scheme, i.e. a scheme that relates four neighbouring points

$$\psi_{m+1,n+1} = \alpha_{m,n}\psi_{m+1,n} + \beta_{m,n}\psi_{m,n+1} + \gamma_{m,n}\psi_{m,n} \tag{1}$$

(where $\alpha,\ \beta,\ \gamma$ are real functions of two discrete variables $(m,n) \in \mathbb{Z}^2$ and where the standard notation $f_{m,n} = f(m,n)$ is used throughout the paper), which is proper for discretizing second order hyperbolic equations in the canonical form. In particular, the proper discrete analogue of the Laplace equation for conjugate nets

$$\Psi_{,uv} + C\Psi_{,u} + D\Psi_{,v} = 0, \tag{2}$$

whose DTs were obtained in [4, 5], turns out to be the 4-point scheme [6]

$$\psi_{m+1,n+1} = \alpha_{m,n}\psi_{m+1,n} + \beta_{m,n}\psi_{m,n+1} + (1-\alpha_{m,n}-\beta_{m,n})\psi_{m,n}, \tag{3}$$

describing a lattice with planar quadrilaterals [7], [8], whose general DTs were extensively studied in recent years (see, e.g., [9, 10, 11]). While the Moutard equation [12]

$$\Psi_{,uv} = F\Psi, \tag{4}$$

relevant in the description of asymptotic nets, splits naturally in the discrete case into two equations [13, 14]

$$\begin{aligned}\psi_{m+1,n+1}+\psi_{m,n}&=f_{m,n}(\psi_{m+1,n}+\psi_{m,n+1})\\ \psi_{m+1,n+1}+\psi_{m,n}&=f_{m+1,n}\psi_{m+1,n}+f_{m,n+1}\psi_{m,n+1}\end{aligned}\tag{5}$$

admitting DTs. Laplace transformations for the general 4-point scheme (1) are also known [7, 15, 16].

The discretization of elliptic equations in two dimensions admitting general DTs is much less studied and understood. In the continuous case, elliptic equations and their Darboux transformations are often obtained from the hyperbolic ones regarding (u, v) as complex variables:

$$u = x + iy \qquad v = x - iy,\tag{6}$$

but this approach presents some problems in the discrete case.

In our opinion, a proper discretization of an elliptic operator should satisfy two basic properties.

- It should be applicable to solve generic Dirichlet boundary value problems on a 2D lattice;

- It should possess a class of DTs which must be (at least) as rich as that of its differential counterpart.

It is very easy to convince one-self that the first criterion cannot be satisfied by the 4-point scheme (1) and that at least 5-point difference schemes should be introduced [17]. The most general 5-point scheme reads as follows:

$$a_{m,n}\psi_{m+1,n}+(a_{m-1,n}+w_{m,n})\psi_{m-1,n}+b_{m,n}\psi_{m,n+1}+(b_{m,n-1}+z_{m,n})\psi_{m,n-1}= f_{m,n}\psi_{m,n},\tag{7}$$

where $a,\ b,\ w,\ z$ are functions of the discrete variables $(m, n) \in \mathbb{Z}^2$.

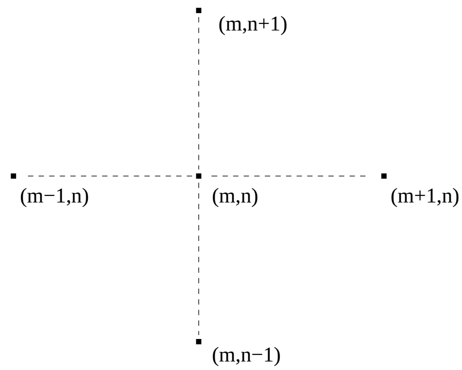

Fig.1 The 5 - point stencil

Its natural continuous limit

$$(\epsilon m, \epsilon n) \ \rightarrow \ (x, y), \tag{8}$$

in which the lattice spacing ϵ goes to zero and

$$\begin{pmatrix} \psi_{m,n} \\ a_{m,n} \\ b_{m,n} \end{pmatrix} = \begin{pmatrix} \Psi(x,y) \\ A(x,y) \\ B(x,y) \end{pmatrix} + O(\epsilon^3), \ \ \begin{pmatrix} w_{m,n} \\ z_{m,n} \end{pmatrix} = -\epsilon \begin{pmatrix} W(x,y) \\ Z(x,y) \end{pmatrix} + O(\epsilon^3), \tag{9}$$

$$f_{m,n} = 2(A+B) - \epsilon(A_{,x} + B_{,y} + W + Z) + \epsilon^2[F(x,y) + \frac{1}{2}(A_{,xx} + B_{,yy})] + O(\epsilon^3),$$

gives the following second order PDE in the independent variables (x, y)

$$(A\Psi_{,x})_{,x} + W\Psi_{,x} + (B\Psi_{,y})_{,y} + Z\Psi_{,y} = F\Psi, \tag{10}$$

in which the $\Psi_{,xy}$ term is missing (we remark that the $\Psi_{,xy}$ term is instead the only second order term present in equations (2) and (4)). Equation (10) is elliptic if $AB > 0$ and its canonical form is obtained by setting $A = 1 = B$. The functions w, z, W, Z measure the departure of equations (7) and (10) from self-adjointness, which is a basic property in most of the applications.

In this paper we present the following results.

- We construct the DTs for the self-adjoint reductions of the 5-point scheme (7) and of its continuous limit (10).

- We use the gauge covariance property of the general self-adjoint 5-point scheme to construct a distinguished integrable discrete analogue of the

stationary Schrödinger operator in two dimensions. "Integrable" in the sense that, not only it reduces to the stationary Schrödinger operator in the continuous limit, but also it possesses DTs.

- We show that the construction of DTs for the self-adjoint 5-point scheme can be applied to the case of self-adjoint schemes involving more neighbouring points, illustrating such a generalization in the case of the general self-adjoint 7-point scheme, for which Laplace transformations are already known [18, 15].

The paper is organized as follows. Section 2 is devoted to the construction of the DTs for the most general self-adjoint 5-point difference operator with complex coefficients:

$$\mathcal{L}_5 = a_{m,n}T_m + a_{m-1,n}T_m^{-1} + b_{m,n}T_n + b_{m,n-1}T_n^{-1} - f_{m,n}, \tag{11}$$

where T_m and T_n are the translation operators with respect to the discrete variables $(m,n) \in \mathbb{Z}^2$:

$$T_m f_{m,n} = f_{m+1,n}, \quad T_n f_{m,n} = f_{m,n+1}$$

and $a,\ b,\ c$ are complex functions of the two discrete variables $(m,n) \in \mathbb{Z}^2$.

We also use the following difference operators

$$\begin{array}{ll} \Delta_m f_{m,n} = f_{m+1,n} - f_{m,n} & \Delta_n f_{m,n} = f_{m,n+1} - f_{m,n} \\ \Delta_{-m} f_{m,n} = f_{m-1,n} - f_{m,n} & \Delta_{-n} f_{m,n} = f_{m,n-1} - f_{m,n}. \end{array}$$

The operator $\mathcal{L}_5$ is formally self-adjoint with respect to the bilinear form:

$$< f, g >:= \sum_{m,n} f_{m,n} g_{m,n} \tag{12}$$

and, in the particular case $a = b = 1$, it reduces to the 5-point scheme

$$\mathcal{L}_{Sch} = T_m + T_m^{-1} + T_n + T_n^{-1} - f, \tag{13}$$

which is the simplest and most used (in numerical applications) discretization of the Schrödinger operator

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - F, \tag{14}$$

while the further constraint $f = 4$ leads to

$$\mathcal{L}_0 = T_m + T_m^{-1} + T_n + T_n^{-1} - 4, \tag{15}$$

i.e. to a discretization of the Laplacian $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ often used in the numerical studies of elliptic boundary value problems [17].

In the continuous limit, the operator $\mathcal{L}_5$ goes to the second order partial differential operator

$$L = A\frac{\partial^2}{\partial x^2} + B\frac{\partial^2}{\partial y^2} + A_{,x}\frac{\partial}{\partial x} + B_{,y}\frac{\partial}{\partial y} - F \tag{16}$$

which is formally self-adjoint with respect to the bilinear form:

$$< f, g >:= \int fg dx dy. \tag{17}$$

We notice that, while both operators (11), (16) are self-adjoint, the two "integrable" discretizations (5) of the self-adjoint Moutard equation (4) are just mutually adjoint.

In section 3, using the following covariance property (gauge invariance) of the operator $\mathcal{L}_5$:

$$\begin{gathered} \mathcal{L}_5 \to \tilde{\mathcal{L}}_5 = g_{m,n}\mathcal{L}_5 g_{m,n} \\ a_{m,n} \to \tilde{a}_{m,n} = a_{m,n}g_{m,n}g_{m+1,n}, \qquad b_{m,n} \to \tilde{b}_{m,n} = b_{m,n}g_{m,n}g_{m,n+1}, \\ f_{m,n} \to \tilde{f}_{m,n} = f_{m,n}g^2_{m,n}, \end{gathered} \tag{18}$$

we draw the operator $\mathcal{L}_5$ to the form

$$\mathcal{L}_{SchInt} = \frac{\Gamma_{m,n}}{\Gamma_{m+1,n}}T_m + \frac{\Gamma_{m-1,n}}{\Gamma_{m,n}}T_m^{-1} + \frac{\Gamma_{m,n}}{\Gamma_{m,n+1}}T_n + \frac{\Gamma_{m,n-1}}{\Gamma_{m,n}}T_n^{-1} - f. \tag{19}$$

The operator (19), symmetric with respect to $\pi/2$ rotations in the plane grid (m, n), reduces to the two-dimensional Schrödinger operator (14) under the natural continuous limit and possesses DTs. Therefore it appears as the "integrable" discrete analogue of the two-dimensional Schrödinger operator. It is therefore the proper starting point in the search for the integrable discrete analogues of the nonlinear PDEs of elliptic type associated with (14) (like the Veselov-Novikov hierarchy [19], nonlinear σ models [20] and the Ernst equation [21]).

The corresponding gauge transformation for the operator L in (16) is

$$\begin{gathered} L \to \tilde{L} = gLg, \\ \tilde{A} = g^2 A, \qquad \tilde{B} = g^2 B, \\ \tilde{F} = gLg. \end{gathered} \tag{20}$$

We draw the reader's attention to the fact that constraint $A = B$ is conserved by the gauge transformation (20) and the same is true for the constraint $a_{m,n+1}a_{m,n} = b_{m+1,n}b_{m,n}$ with respect to gauge (18). In the case of the reduction $A = B$, the operator (16) can be gauged into the form of the 2D Schrödinger operator (14) while, in the case $a_{m,n+1}a_{m,n} = b_{m+1,n}b_{m,n}$, the operator (11) can be gauged into the operator (13). But while there exist DTs which preserves the form of equation (14), we don't know DTs that preserves the form of equation (13). This is the reason why we view the operator (19) and not the operator (13) as the integrable discretization of the Schrödinger equation (14).

Finally, in section 4 we show that there is another self-adjoint scheme involving more neighbouring points which possess DTs of the type presented in section 2. It is a self-adjoint 7-point difference scheme

$$\begin{aligned} &\mathcal{L}_7\psi_{m,n} := a_{m,n}\psi_{m+1,n} + a_{m-1,n}\psi_{m-1,n} + b_{m,n}\psi_{m,n+1} + \\ &b_{m,n-1}\psi_{m,n-1} + s_{m+1,n}\psi_{m+1,n-1} + s_{m,n+1}\psi_{m-1,n+1} - f_{m,n}\psi_{m,n} = 0, \end{aligned} \tag{21}$$

which, in the continuous limit

$$\begin{pmatrix} \psi_{m,n} \\ a_{m,n} + s_{m,n} \\ b_{m,n} + s_{m,n} \\ s_{m,n} \end{pmatrix} = \begin{pmatrix} \Psi(x,y) \\ A(x,y) \\ B(x,y) \\ -S(x,y) \end{pmatrix} + O(\epsilon^3), \tag{22}$$

$$f_{m,n} = 2(A+B+S) - \epsilon(A_{,x} + B_{,y} + 2S_{,x} + 2S_{,y}) + \epsilon^2[F(x,y) + \frac{1}{2}(A_{,xx} + B_{,yy})] + O(\epsilon^3)$$

goes to the most general second order self-adjoint differential equation in two independent variables:

$$(A\Psi_{,x})_{,x} + (S\Psi_{,y})_{,x} + (B\Psi_{,y})_{,y} + (S\Psi_{,x})_{,y} = F\Psi. \tag{23}$$

The operator $\mathcal{L}_7$ in (21), introduced in [18], admits the representation $\mathcal{L}_7 = QQ^+ + w$ in terms of a 3-point difference operator Q and of its adjoint

Q^+, and this factorization plays a crucial role in the construction of Laplace transformations for $\mathcal{L}_7$ [18, 15] and in the development of an associated discrete complex function theory [22]. The 5-point operator $\mathcal{L}_5$ does not admit the above representation and the construction of the DTs of this paper does not make use of it. We also remark that several discretizations of the 1D Schrödinger operator have appeared in the literature throughout the years (see, f.i., [23, 24, 25] and [15] with references therein included). We end this introduction observing that a 4-point scheme with a complexification of the discrete variables (m, n) analogous to (6), was used in [26] to construct a discrete analogue of the σ model.

2 5-point self adjoint operator and its Darboux transformation

In this section we present a Darboux transformation for the equation:

$$a_{m,n}\psi_{m+1,n} + a_{m-1,n}\psi_{m-1,n} + b_{m,n}\psi_{m,n+1} + b_{m,n-1}\psi_{m,n-1} = f_{m,n}\psi_{m,n}, \quad (24)$$

where $a_{m,n}$, $b_{m,n}$ and $f_{m,n}$ are given functions. Let θ be another solution of (24), i.e:

$$a_{m,n}\theta_{m+1,n} + a_{m-1,n}\theta_{m-1,n} + b_{m,n}\theta_{m,n+1} + b_{m,n-1}\theta_{m,n-1} = f_{m,n}\theta_{m,n}; \quad (25)$$

then:

$$f_{m,n} = \frac{1}{\theta}\left(a_{m,n}\theta_{m+1,n} + a_{m-1,n}\theta_{m-1,n} + b_{m,n}\theta_{m,n+1} + b_{m,n-1}\theta_{m,n-1}\right). \quad (26)$$

Eliminating $f_{m,n}$ from (24) and (25) we get

$$\begin{array}{l} \Delta_m(a_{m-1,n}\psi_{m,n}\theta_{m-1,n} - a_{m-1,n}\theta_{m,n}\psi_{m-1,n}) + \\ \Delta_n(b_{m,n-1}\psi_{m,n}\theta_{m,n-1} - b_{m,n-1}\theta_{m,n}\psi_{m,n-1}) = 0. \end{array} \quad (27)$$

It means that there exists a function α such that

$$\begin{array}{l} \Delta_n\alpha = a_{m-1,n}\theta_{m,n}\theta_{m-1,n}\Delta_{-m}\frac{\psi_{m,n}}{\theta_{m,n}}, \\ \Delta_m\alpha = -b_{m,n-1}\theta_{m,n}\theta_{m,n-1}\Delta_{-n}\frac{\psi_{m,n}}{\theta_{m,n}}. \end{array} \quad (28)$$

Setting

$$\psi'_{m,n} = \frac{\alpha_{m,n}}{\theta_{m,n}}$$

we find that $\psi'_{m,n}$ satisfies the following equation

$$a'_{m,n}\,\psi'_{m+1,n} + a'_{m-1,n}\,\psi'_{m-1,n} + b'_{m,n}\,\psi'_{m,n+1} + b'_{m,n-1}\,\psi'_{m,n-1} = f'_{m,n}\,\psi'_{m,n}, \quad (29)$$

where

$$a'_{m-1,n} = \frac{\theta_{m,n}}{b_{m-1,n-1}\,\theta_{m-1,n-1}} \qquad b'_{m,n-1} = \frac{\theta_{m,n}}{a_{m-1,n-1}\,\theta_{m-1,n-1}} \quad (30)$$

and

$$f'_{m,n} = \theta_{m,n}\left(a'_{m,n}\,\frac{1}{\theta_{m+1,n}} + a'_{m-1,n}\,\frac{1}{\theta_{m-1,n}} + b'_{m,n}\,\frac{1}{\theta_{m,n+1}} + b'_{m,n-1}\,\frac{1}{\theta_{m,n-1}}\right). \quad (31)$$

Comparing equations (26) and (31), we also infer that $\theta' = 1/\theta$ is a solution of (29).

In the continuous limit (8), with $\theta_{m,n} = \Theta(x,y) + O(\epsilon^3)$ and with $f_{m,n}$ expanded according to (26) to get (9b), we obtain the DT

$$\frac{1}{\Theta}(\Theta\Psi')_{,y} = -A\Theta(\frac{\Psi}{\Theta})_{,x} \qquad \frac{1}{\Theta}(\Theta\Psi')_{,x} = B\Theta(\frac{\Psi}{\Theta})_{,y} \quad (32)$$

from the solution space of equation

$$(A\Psi,{}_{,x})_{,x} + (B\Psi_{,y})_{,y} = F\Psi \quad (33)$$

to the solution space of equation

$$(A'\Psi'_{,x})_{,x} + (B'\Psi'_{,y})_{,y} = F'\Psi', \quad (34)$$

where Θ is another solution of (33), so that

$$F = \frac{1}{\Theta}[(A\Theta_{,x})_{,x} + (B\Theta_{,y})_{,y}], \quad (35)$$

and the new "potentials" A', B' and F' are related to the old ones as follows

$$\begin{gathered} A' = \tfrac{1}{B}, \qquad B' = \tfrac{1}{A}, \\ F' = \Theta[(A'(\tfrac{1}{\Theta})_{,x})_{,x} + (B'(\tfrac{1}{\Theta})_{,y})_{,y}]. \end{gathered} \quad (36)$$

Comparing equations (34) and (36c), we also infer that $\Theta' = 1/\Theta$ is a solution of (34).

3 A two-dimensional Schrödinger operator and its Darboux transformation

The operator (11) can be gauged into the form (19). Indeed, if $a_{m,n}$ and $b_{m,n}$ are given functions, one can find a function $g_{m,n}$ such that $\frac{a_{m,n}}{b_{m,n}} = \frac{g_{m,n+1}}{g_{m+1,n}}$ and, under the gauge (18), we get $\tilde{a}_{m,n} = \tilde{b}_{m,n} =: \frac{1}{\Gamma^2_{m,n}}$. Finally, the operator $\hat{\mathcal{L}}$ given by $\hat{\mathcal{L}}_5 = \frac{1}{\Gamma_{m,n}}\tilde{\mathcal{L}}_5\frac{1}{\Gamma_{m,n}}$ is of the wanted form (19).

Combining the DT of the previous section with the above gauge transformation, one obtains the following DT for the discrete analogue (19) of the Schrödinger operator.

Let $N_{m,n}$ be a solution of the integrable discrete analogue of the 2D Schrödinger operator:

$$\frac{\Gamma_{m,n}}{\Gamma_{m+1,n}}N_{m+1,n} + \frac{\Gamma_{m-1,n}}{\Gamma_{m,n}}N_{m-1,n} + \frac{\Gamma_{m,n}}{\Gamma_{m,n+1}}N_{m,n+1} + \frac{\Gamma_{m,n-1}}{\Gamma_{m,n}}N_{m,n-1} = f_{m,n}N_{m,n}, \tag{37}$$

and let ϑ be another solution of it. It means that

$$f_{m,n} = \frac{1}{\vartheta}\left(\frac{\Gamma_{m,n}}{\Gamma_{m+1,n}}\vartheta_{m+1,n} + \frac{\Gamma_{m-1,n}}{\Gamma_{m,n}}\vartheta_{m-1,n} + \frac{\Gamma_{m,n}}{\Gamma_{m,n+1}}\vartheta_{m,n+1} + \frac{\Gamma_{m,n-1}}{\Gamma_{m,n}}\vartheta_{m,n-1}\right). \tag{38}$$

Eliminating $f_{m,n}$ from (38) and (37) we get that there exists a function $N'_{m,n}$, given in quadratures by

$$\begin{aligned} \Delta_m\left(\frac{\vartheta_{m,n}}{\Gamma_{m,n}}\Gamma'_{m,n}N'_{m,n}\right) &= \frac{\Gamma_{m,n-1}}{\Gamma_{m,n}}\vartheta_{m,n}\vartheta_{m,n-1}\left(\frac{N_{m,n}}{\vartheta_{m,n}} - \frac{N_{m,n-1}}{\vartheta_{m,n-1}}\right)\\ \Delta_n\left(\frac{\vartheta_{m,n}}{\Gamma_{m,n}}\Gamma'_{m,n}N'_{m,n}\right) &= -\frac{\Gamma_{m-1,n}}{\Gamma_{m,n}}\vartheta_{m,n}\vartheta_{m-1,n}\left(\frac{N_{m,n}}{\vartheta_{m,n}} - \frac{N_{m-1,n}}{\vartheta_{m-1,n}}\right) \end{aligned} \tag{39}$$

which satisfies

$$\frac{\Gamma'_{m,n}}{\Gamma'_{m+1,n}}N'_{m+1,n} + \frac{\Gamma'_{m-1,n}}{\Gamma'_{m,n}}N'_{m-1,n} + \frac{\Gamma'_{m,n}}{\Gamma'_{m,n+1}}N'_{m,n+1} + \frac{\Gamma'_{m,n-1}}{\Gamma'_{m,n}}N'_{m,n-1} = f'_{m,n}N'_{m,n}, \tag{40}$$

where

$$\begin{aligned} &\Gamma'^2_{m,n} = \frac{\Gamma_{m,n}\Gamma_{m-1,n-1}\vartheta_{m-1,n-1}}{\vartheta_{m,n}},\\ &f'_{m,n} = \frac{\Gamma'\vartheta}{\Gamma}\left(\frac{\Gamma'_{m,n}}{\Gamma'_{m+1,n}}(\tfrac{\Gamma}{\Gamma'\vartheta})_{m+1,n} + \frac{\Gamma'_{m-1,n}}{\Gamma'_{m,n}}(\tfrac{\Gamma}{\Gamma'\vartheta})_{m-1,n} + \frac{\Gamma'_{m,n}}{\Gamma'_{m,n+1}}(\tfrac{\Gamma}{\Gamma'\vartheta})_{m,n+1} +\right.\\ &\left.\frac{\Gamma'_{m,n-1}}{\Gamma'_{m,n}}(\tfrac{\Gamma}{\Gamma'\vartheta})_{m,n-1}\right) = (\Gamma\vartheta)_{m-1,n-1}[\tfrac{1}{(\Gamma\vartheta)_{m-1,n}} + \tfrac{1}{(\Gamma\vartheta)_{m,n-1}}] + \tfrac{\vartheta}{\Gamma}[(\tfrac{\Gamma}{\vartheta})_{m-1,n} + (\tfrac{\Gamma}{\vartheta})_{m,n-1}]. \end{aligned} \tag{41}$$

Now the function $\vartheta' = \frac{\Gamma}{\Gamma'\vartheta}$ is a solution of (40). In the continuous limit, with $\Gamma_{m,n} = G(x,y) + O(\epsilon)$ and with $f_{m,n}$ expanded according to equation (38), one obtains the classical Moutard transformation

$$\frac{1}{\Theta}(\Theta N')_{,x} = \Theta\left(\frac{N}{\Theta}\right)_{,y} \;, \qquad \frac{1}{\Theta}(\Theta N')_{,y} = -\Theta\left(\frac{N}{\Theta}\right)_{,x} \tag{42}$$

from the solution space of equation

$$\Psi_{,xx} + \Psi_{,yy} = F\Psi \tag{43}$$

to the solution space of equation

$$\Psi'_{,xx} + \Psi'_{,yy} = F'\Psi', \tag{44}$$

where Θ is another solution of (43), so that

$$F = \frac{1}{\Theta}(\Theta_{,xx} + \Theta_{,yy}),$$

and

$$F' = \Theta[\left(\frac{1}{\Theta}\right)_{,xx} + \left(\frac{1}{\Theta}\right)_{,yy}].$$

Furthermore $\Theta' = 1/\Theta$ is a solution of (44).

4 7-point self-adjoint operator and its Darboux transformations

We end this paper showing that the construction of DTs presented in the previous two sections applies also to a self-adjoint scheme involving more than 5 points, namely in the case of the self-adjoint 7-point scheme:

$$\begin{aligned} &a_{m,n}\psi_{m+1,n} + a_{m-1,n}\psi_{m-1,n} + b_{m,n}\psi_{m,n+1} + b_{m,n-1}\psi_{m,n-1} + \\ &s_{m+1,n}\psi_{m+1,n-1} + s_{m,n+1}\psi_{m-1,n+1} = f_{m,n}\psi_{m,n}, \end{aligned} \tag{45}$$

where $a_{m,n}$, $b_{m,n}$, $s_{m,n}$ and $f_{m,n}$ are given functions, which, as we have seen in the introduction, is a discretization of the most general second order, self-adjoint, linear, differential equation in two independent variables.

Let $\theta_{m,n}$ be another solution of equation (45):

$$
\begin{array}{l}
a_{m,n}\theta_{m+1,n} + a_{m-1,n}\theta_{m-1,n} + b_{m,n}\theta_{m,n+1} + b_{m,n-1}\theta_{m,n-1} + \\
s_{m+1,n}\theta_{m+1,n-1} + s_{m,n+1}\theta_{m-1,n+1} = f_{m,n}\theta_{m,n}.
\end{array} \tag{46}
$$

Eliminating $f_{m,n}$ from (45) and (46) we get

$$
\begin{array}{l}
\Delta_m[a_{m-1,n}\theta_{m,n}\theta_{m-1,n}(\frac{\psi_{m,n}}{\theta_{m,n}} - \frac{\psi_{m-1,n}}{\theta_{m-1,n}}) + s_{m,n}\theta_{m-1,n}\theta_{m,n-1}(\frac{\psi_{m,n-1}}{\theta_{m,n-1}} - \frac{\psi_{m-1,n}}{\theta_{m-1,n}})] + \\
\Delta_n[b_{m,n-1}\theta_{m,n}\theta_{m,n-1}(\frac{\psi_{m,n}}{\theta_{m,n}} - \frac{\psi_{m,n-1}}{\theta_{m,n-1}}) + s_{m,n}\theta_{m-1,n}\theta_{m,n-1}(\frac{\psi_{m-1,n}}{\theta_{m-1,n}} - \frac{\psi_{m,n-1}}{\theta_{m,n-1}})] = 0.
\end{array} \tag{47}
$$

It means that there exists a function α such that

$$
\begin{array}{l}
\Delta_n\alpha_{m,n} = (a_{m-1,n}\theta_{m,n}\theta_{m-1,n} + s_{m,n}\theta_{m-1,n}\theta_{m,n-1})\,\Delta_{-m}\frac{\psi_{m,n}}{\theta_{m,n}} - s_{m,n}\theta_{m-1,n}\theta_{m,n-1}\Delta_{-n}\frac{\psi_{m,n}}{\theta_{m,n}} \\
\Delta_m\alpha_{m,n} = -(b_{m,n-1}\theta_{m,n}\theta_{m,n-1} + s_{m,n}\theta_{m-1,n}\theta_{m,n-1})\Delta_{-n}\frac{\psi_{m,n}}{\theta_{m,n}} + s_{m,n}\theta_{m-1,n}\theta_{m,n-1}\Delta_{-m}\frac{\psi_{m,n}}{\theta_{m,n}}.
\end{array} \tag{48}
$$

Introducing

$$
\psi'_{m,n} = \frac{\alpha_{m,n}}{\theta_{m,n}}
$$

we find that $\psi'_{m,n}$ satisfies the following equation

$$
\begin{array}{l}
a'_{m,n}\,\psi'_{m+1,n} + a'_{m-1,n}\,\psi'_{m-1,n} + b'_{m,n}\,\psi'_{m,n+1} + b'_{m,n-1}\,\psi'_{m,n-1} + \\
s'_{m+1,n}\psi'_{m+1,n-1} + s'_{m,n+1}\psi'_{m-1,n+1} = f'_{m,n}\,\psi'_{m,n},
\end{array} \tag{49}
$$

where the new fields are given by

$$
\begin{array}{lcl}
a'_{m,n} & = & \frac{\theta_{m,n}\theta_{m+1,n}a_{m-1,n}}{\theta_{m,n-1}p_{m,n}}, \\
b'_{m,n} & = & \frac{\theta_{m,n}\theta_{m,n+1}b_{m,n-1}}{\theta_{m-1,n}p_{m,n}}, \\
s'_{m,n} & = & \frac{s_{m-1,n-1}\theta_{m-1,n}\theta_{m,n-1}}{\theta_{m-1,n-1}p_{m-1,n-1}}, \\
f'_{m,n} & = & \theta_{m,n}(a'_{m,n}\frac{1}{\theta_{m+1,n}} + a'_{m-1,n}\frac{1}{\theta_{m-1,n}} + b'_{m,n}\frac{1}{\theta_{m,n+1}} + b'_{m,n-1}\frac{1}{\theta_{m,n-1}} + \\
 & & s'_{m+1,n}\frac{1}{\theta_{m+1,n-1}} + s'_{m,n+1}\frac{1}{\theta_{m-1,n+1}})
\end{array} \tag{50}
$$

and where $p_{m,n} = \theta_{m,n}a_{m-1,n}b_{m,n-1} + \theta_{m-1,n}s_{m,n}a_{m-1,n} + s_{m,n}\theta_{m,n-1}b_{m,n-1}$. Again $\theta'_{m,n} = 1/\theta_{m,n}$ is a solution of (49).

In the continuous limit (22), with $\theta_{m,n} = \Theta(x,y) + O(\epsilon^3)$ and with $f_{m,n}$ expanded according to (46) to obtain (22), we obtain the DT

$$
\begin{array}{c}
\frac{1}{\Theta}(\Theta\Psi')_{,y} = -A\Theta(\frac{\Psi}{\Theta})_{,x} - S\Theta(\frac{\Psi}{\Theta})_{,y} \\
\frac{1}{\Theta}(\Theta\Psi')_{,x} = B\Theta(\frac{\Psi}{\Theta})_{,y} + S\Theta(\frac{\Psi}{\Theta})_{,x}
\end{array} \tag{51}
$$

from the solution space of equation

$$(A\Psi_{,x})_{,x}+(B\Psi_{,y})_{,y}+(S\Psi_{,y})_{,x}+(S\Psi_{,x})_{,y}=F\Psi \tag{52}$$

to the solution space of equation

$$(A'\Psi'_{,x})_{,x}+(B'\Psi'_{,y})_{,y}+(S'\Psi'_{,y})_{,x}+(S'\Psi'_{,x})_{,y}=F'\Psi', \tag{53}$$

where Θ is fixed solution of (52); so

$$F=\frac{1}{\Theta}[(A\Theta_{,x})_{,x}+(B\Theta_{,y})_{,y}+(S\Theta_{,y})_{,x}+(S\Theta_{,x})_{,y}] \tag{54}$$

and the new "potentials" A', B', S' and F' are related to the old ones as follows

$$\begin{gathered} A'=\tfrac{A}{AB-S^2}, \qquad B'=\tfrac{B}{AB-S^2}, \qquad S'=\tfrac{S}{AB-S^2}, \\ F'=\Theta[(A'(\tfrac{1}{\Theta})_{,x})_{,x}+(B'(\tfrac{1}{\Theta})_{,y})_{,y}+(S'(\tfrac{1}{\Theta})_{,x})_{,y}+(S'(\tfrac{1}{\Theta})_{,y})_{,x}]. \end{gathered} \tag{55}$$

Again $\Theta'=1/\Theta$ is a solution of (53).

Acknowledgments One of us (PMS) acknowledges useful discussions with M.J.Ablowitz. This work was supported by the cultural and scientific agreement between the University of Roma "La Sapienza" and the University of Warsaw and by the University of Warmia and Mazury in Olsztyn under the grant 522-1307-0201 and partially supported by KBN grant 2 P03B 126 22.

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